Matrix and Determinant

1. Use the Crammer's rule to discuss the consistency of the following system of equation for different cases of λ:

$$\begin{cases} x + y + \lambda z = 1 \\ x + \lambda y + z = \lambda \\ \lambda x + y + z = \lambda^2 \end{cases}$$

2. Let $A = \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ where $\alpha \neq 1$.

(a) Prove that for all positive integer n,

$$A^n = \begin{pmatrix} \alpha^n & \dfrac{\beta(\alpha^n - 1)}{\alpha - 1} \\ 0 & 1 \end{pmatrix}$$

(b) **(i)** Find BA.

(ii) Use **(a)** , or otherwise, evaluate $(BA)^n$, for $n \in \mathbf{N}$**,**

3. **(a)** Factorize $F(\alpha,\beta,\gamma) = \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix}$.

(b) Show that $\begin{vmatrix} (1+ax)^2 & (1+bx)^2 & (1+cx)^2 \\ (1+ay)^2 & (1+by)^2 & (1+cy)^2 \\ (1+az)^2 & (1+bz)^2 & (1+cz)^2 \end{vmatrix} = k\, F(x,y,z)\, F(a,b,c)$

where k is a constant to be found, and hence factorize the determinant.

4. If n is the least positive integer such that A^n is a zero matrix, then A is said to be nilpotent of order n.

Given A is a nilpotent of order n.

(a) **(i)** Evaluate $(I - A)\left(I + A + A^2 + \ldots + A^{n-1}\right)$ and

$$(I + A)\left\{I - A + A^2 - \ldots + (-1)^{n-1} A^{n-1}\right\}$$

(ii) Hence, or otherwise, express $(I - A)^{-1}$ and $(I - A^2)^{-1}$ in terms of A.

(b) Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

(i) Evaluate A^2 and A^3.

(ii) Using (a), or otherwise, find $(I - A)^{-1}$ and $(I - A^2)^{-1}$.

5. Let $A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & -2 & 1 \\ 1 & 0 & 2 \end{pmatrix}$

(a) Find $A^3 - 2A^2 - 7A + I$ where I is the identity matrix of order 3 x 3.

(b) Using (a), evaluate $(A - I)(A^2 - A - 8I)$.

(c) Hence find $(A^2 - A - 8I)^{-1}$.

6. Find the equation of the image of the curve :

$$5x^2 - 2\sqrt{3}xy + 7y^2 - 4 = 0$$

if the **curve** is under the rotation transformation through an angle $\frac{5\pi}{6}$ anti-clockwisely about the origin.

Hint : The formula for rotating anti-clockwisely by an angle θ is

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

1. $\Delta = \begin{vmatrix} 1 & 1 & \lambda \\ 1 & \lambda & 1 \\ \lambda & 1 & 1 \end{vmatrix} = -(\lambda-1)^2(\lambda+2)$

$\Delta_x = \begin{vmatrix} 1 & 1 & \lambda \\ \lambda & \lambda & 1 \\ \lambda^2 & 1 & 1 \end{vmatrix} = -(\lambda+1)^2(\lambda-1)^2, \quad \Delta_y = \begin{vmatrix} 1 & 1 & \lambda \\ 1 & \lambda & 1 \\ \lambda & \lambda^2 & 1 \end{vmatrix} = -(\lambda-1)^2, \quad \Delta_z = \begin{vmatrix} 1 & 1 & \lambda \\ 1 & \lambda & 1 \\ \lambda & 1 & 1 \end{vmatrix} = (\lambda+1)(\lambda-1)^2$

Conclusions :

(i) If $\lambda \neq 1, \ -2, \ \Delta \neq 0$. $\therefore$ The system has unique solution:

$$(x, y, z) = \left(\frac{(\lambda+1)^2}{\lambda+2}, \ \frac{1}{\lambda+2}, \ \frac{\lambda+1}{\lambda+2} \right).$$

(ii) If $\lambda = 1$, $\Delta = \Delta_x = \Delta_y = \Delta_z$ and the system of equation becomes $x + y + z = 1$.

$\therefore$ The system of equation has infinitely many solutions:

$(x, y, z) = (\lambda_1, \lambda_2, 1 - \lambda_1 - \lambda_2)$.

(iii) If $\lambda = -2$, $\Delta = 0$, $\Delta_x \neq 0$, $\Delta_y \neq 0$, $\Delta_z \neq 0$.

$\therefore$ The system is inconsistent and has no solution.

2. **(a)** Let P(n) be the proposition: $A^n = \begin{pmatrix} \alpha^n & \dfrac{\beta(\alpha^n - 1)}{\alpha - 1} \\ 0 & 1 \end{pmatrix}$

For P(1), $A^1 = \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha^1 & \dfrac{\beta(\alpha^1 - 1)}{\alpha - 1} \\ 0 & 1 \end{pmatrix}$. $\therefore$ P(1) is true.

Assume P(k) is true for some $k \in N$, i.e. $A^k = \begin{pmatrix} \alpha^k & \dfrac{\beta(\alpha^k - 1)}{\alpha - 1} \\ 0 & 1 \end{pmatrix}$ (1)

For P(k + 1), $A^{k+1} = A^k \cdot A = \begin{pmatrix} \alpha^k & \dfrac{\beta(\alpha^k - 1)}{\alpha - 1} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}$, by (1).

$$= \begin{pmatrix} \alpha^{k+1} & \alpha^k\beta + \dfrac{\beta(\alpha^k - 1)}{\alpha - 1} \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha^{k+1} & \dfrac{\alpha^k\beta(\alpha-1) + \beta(\alpha^k - 1)}{\alpha - 1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha^{k+1} & \dfrac{\beta(\alpha^{k+1} - 1)}{\alpha - 1} \\ 0 & 1 \end{pmatrix}.$$

$\therefore$ P(k + 1) is true.

(b) $BA = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \beta+1 \\ 0 & 1 \end{pmatrix}$ $\therefore$ $(BA)^n = \begin{pmatrix} \alpha & \beta+1 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} \alpha^n & \dfrac{(\beta+1)(\alpha^n - 1)}{\alpha - 1} \\ 0 & 1 \end{pmatrix}$

3. **(a)** $F(\alpha,\beta,\gamma)=\begin{vmatrix}1&1&1\\ \alpha&\beta&\gamma\\ \alpha^2&\beta^2&\gamma^2\end{vmatrix}\begin{matrix}R_2\to R_2-\alpha R_1\\ =\\ R_3\to R_3-\alpha R_2\end{matrix}\begin{vmatrix}1&1&1\\ 0&\beta-\alpha&\gamma-\alpha\\ 0&\beta(\beta-\alpha)&\gamma(\gamma-\alpha)\end{vmatrix}=\begin{vmatrix}\beta-\alpha&\gamma-\alpha\\ \beta(\beta-\alpha)&\gamma(\gamma-\alpha)\end{vmatrix}$

$=(\beta-\alpha)(\gamma-\alpha)\begin{vmatrix}1&1\\ \beta&\gamma\end{vmatrix}=(\beta-\alpha)(\gamma-\alpha)(\gamma-\beta)=(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)$

(b) **Method 1**

$$\begin{vmatrix}(1+ax)^2&(1+bx)^2&(1+cx)^2\\ (1+ay)^2&(1+by)^2&(1+cy)^2\\ (1+az)^2&(1+bz)^2&(1+cz)^2\end{vmatrix}=\begin{vmatrix}1+2ax+a^2x^2&1+2bx+b^2x^2&1+2cx+c^2x^2\\ 1+2ay+a^2y^2&1+2by+b^2y^2&1+2cy+c^2y^2\\ 1+2az+a^2z^2&1+2bz+b^2z^2&1+2cz+c^2z^2\end{vmatrix}$$

$$=\begin{vmatrix}1&2x&x^2\\ 1&2y&y^2\\ 1&2z&z^2\end{vmatrix}\begin{vmatrix}1&1&1\\ a&b&c\\ a^2&b^2&c^2\end{vmatrix}=2\begin{vmatrix}1&x&x^2\\ 1&y&y^2\\ 1&z&z^2\end{vmatrix}\begin{vmatrix}1&1&1\\ a&b&c\\ a^2&b^2&c^2\end{vmatrix}$$

$= 2\,F(x, y, z)\,F(a, b, c)$

$= 2(x-y)(y-z)(z-x)(a-b)(b-c)(c-a)$

Method 2

Let Δ be the given determinant.

Put $x = y$ in Δ, Since $R_1 = R_2$, $\Delta = 0$ and $(x-y)$ is a factor of Δ.

Put $y = z$ in Δ, Since $R_2 = R_3$, $\Delta = 0$ and $(y-z)$ is a factor of Δ.

Put $z = x$ in Δ, Since $R_3 = R_1$, $\Delta = 0$ and $(z-x)$ is a factor of Δ.

Put $a = b$ in Δ, Since $C_1 = C_2$, $\Delta = 0$ and $(a-b)$ is a factor of Δ.

Put $b = c$ in Δ, Since $C_2 = C_3$, $\Delta = 0$ and $(b-c)$ is a factor of Δ.

Put $c = a$ in Δ, Since $C_3 = C_1$, $\Delta = 0$ and $(c-a)$ is a factor of Δ.

$\therefore\ \Delta = kF(x, y, z)\,F(a, b, c) = k(x-y)(y-z)(z-x)(a-b)(b-c)(c-a)$.

4. **(a)** **(i)** $(I-A)(I+A+A^2+\dots+A^{n-1}) = I-A^n = I-\mathbf{0} = I$ (since $A^n = \mathbf{0}$)

$(I+A)\{I-A+A^2-\dots+(-1)^{n-1}A^{n-1}\} = I+(-1)^n A^n = I-\mathbf{0} = I$ (since $A^n = \mathbf{0}$)

(ii) $(I-A)^{-1} = I+A+A^2+\dots+A^{n-1}$

$(I+A)^{-1} = I-A+A^2-\dots+(-1)^{n-1}A^{n-1}$

$(I-A^2)^{-1} = (I+A+A^2+\dots+A^{n-1})\,[I-A+A^2-\dots+(-1)^{n-1}A^{n-1}]$

$= I+A^2+A^4+\dots+A^{2(n-1)}$

$=\begin{cases}I+A^2+A^4+\dots+A^{n-2}, & \text{when n is even}\\ I+A^2+A^4+\dots+A^{n-1}, & \text{when n is odd}\end{cases}$

(b) **(i)** $A^2=\begin{pmatrix}0&0&0\\ 3&3&9\\ -1&-1&-3\end{pmatrix}$ $A^3=\begin{pmatrix}0&0&0\\ 0&0&0\\ 0&0&0\end{pmatrix}$

(ii) By (b) (i), A is a nilpotent matrix of order 3.

By (a) (ii), $(I-A)^{-1} = I+A+A^2$

$$=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\begin{pmatrix}2&1&3\\0&-2&1\\1&0&2\end{pmatrix}+\begin{pmatrix}0&0&0\\3&3&9\\-1&-1&-3\end{pmatrix}=\begin{pmatrix}2&1&3\\8&6&15\\-3&-2&-5\end{pmatrix}$$

By (a) (ii), $(I-A^2)^{-1} = I + A^2$

$$=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}+\begin{pmatrix}0&0&0\\3&3&9\\-1&-1&-3\end{pmatrix}=\begin{pmatrix}1&0&0\\3&4&9\\-1&-1&-2\end{pmatrix}$$

5. **(a)** $A^2=\begin{pmatrix}7&0&13\\1&4&0\\4&1&7\end{pmatrix}$, $A^3=\begin{pmatrix}27&7&47\\2&-7&7\\15&2&27\end{pmatrix}$

$\therefore$ A^3-2A^2-7A+I

$$=\begin{pmatrix}27&7&47\\2&-7&7\\15&2&27\end{pmatrix}-2\begin{pmatrix}7&0&13\\1&4&0\\4&1&7\end{pmatrix}-7\begin{pmatrix}2&1&3\\0&-2&1\\1&0&2\end{pmatrix}+\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}$$

(b) $(A-I)(A^2-A-8I) = A^3-2A^2-7A+8I=(A^3-2A^2-7A+I)+7I$, by (a).

$$=\begin{pmatrix}7&0&0\\0&7&0\\0&0&7\end{pmatrix}$$

(c) By (b), $(A^2-A-8I)^{-1} = \dfrac{1}{7}I=\dfrac{1}{7}\begin{pmatrix}1&1&3\\0&-3&1\\1&0&1\end{pmatrix}$

6. The matrix of rotation $=\begin{pmatrix}\cos\frac{5\pi}{6}&-\sin\frac{5\pi}{6}\\\sin\frac{5\pi}{6}&\cos\frac{5\pi}{6}\end{pmatrix}=\dfrac{1}{2}\begin{pmatrix}-\sqrt{3}&-1\\1&-\sqrt{3}\end{pmatrix}$

$$\begin{pmatrix}x'\\y'\end{pmatrix}=\frac{1}{2}\begin{pmatrix}-\sqrt{3}&-1\\1&\sqrt{3}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$\therefore \quad \begin{pmatrix}x\\y\end{pmatrix}=2\begin{pmatrix}-\sqrt{3}&-1\\1&-\sqrt{3}\end{pmatrix}^{-1}\begin{pmatrix}x'\\y'\end{pmatrix}=\frac{1}{2}\begin{pmatrix}-\sqrt{3}&1\\-1&-\sqrt{3}\end{pmatrix}\begin{pmatrix}x'\\y'\end{pmatrix}=\begin{pmatrix}\frac{1}{2}\left(-\sqrt{3}x'+y'\right)\\\frac{1}{2}\left(-x'-\sqrt{3}y'\right)\end{pmatrix}$$

Equation of the image is:

$$5\left[\frac{1}{2}\left(-\sqrt{3}x'+y'\right)\right]^2-2\sqrt{3}\left[\frac{1}{2}\left(-\sqrt{3}x'+y'\right)\right]\left[\frac{1}{2}\left(-x'-\sqrt{3}y'\right)\right]+7\left[\frac{1}{2}\left(-x'-\sqrt{3}y'\right)\right]^2-4=0$$

Simplify, we get $16(x')^2 + 32(y')^2 - 16 = 0$

$(x')^2 + 2(y')^2 = 1$, an ellipse.

Yue Kwok Choy

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